CALCULUS OF VARIATIONS.

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Second Article.

1. *Discussion of the first variation.* As a more general form of the integrals which were given in problems I, II, III, and IV (see Annals of Mathematics, Vol. IX, No. 6, p. 183 et seq.) let us consider the integral

$$I = \int_{x_0}^{x_1} F(x, y; y')\, dx ,$$

where $F(x, y; y)$ is a known function of x, y, and y', and where the limits of this integral, x_1 and x_0, are fixed. Hence (loc. cit., p. 189)

$$\Delta I = \int_{x_0}^{x_1} F(x, y + \varepsilon\eta;\ y' + \varepsilon\eta')\, dx - \int_{x_0}^{x_1} F(x, y;\ y')\, dx$$

$$= \int_{x_0}^{x_1} [F(x, y + \varepsilon\eta;\ y' + \varepsilon\eta') - F(x, y;\ y')]\, dx .$$

This expression, when expanded by Taylor's theorem, is

$$= \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial y}\, \varepsilon\eta + \frac{\partial F}{\partial y'}\, \varepsilon\eta' + \varepsilon^2 (\,) + \dots \right] dx .$$

We also have, as on p. 189,

$$\Delta I = \varepsilon \delta I + \frac{\varepsilon^2}{1\,.\,2}\, \delta^2 I + \dots ;$$

and by comparing the coefficients of ε in these two expressions, it follows that

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial y}\, \eta + \frac{\partial F}{\partial y'}\, \eta' \right] dx . \tag{A}$$

Remark. In the particular case given (p. 183), $F = 2\pi y \sqrt{1 + y'^2}$. Hence

$$\frac{\partial F}{\partial y} = 2\pi \sqrt{1 + y'^2} \text{ and } \frac{\partial F}{\partial y'} = \frac{2\pi y y'}{\sqrt{1 + y'^2}} ;$$

and when these relations are substituted in (A), we have, as on p. 190,

$$\delta I = 2\pi \int_{x_0}^{x_1} \left[\sqrt{1+y'^2}\, \eta + \frac{yy'}{\sqrt{1+y'^2}}\eta' \right] dx .$$

2. From the relation

$$\Delta I = \varepsilon \delta I + \frac{\varepsilon^2}{1\,.\,2}\delta^2 I + \ldots ,$$

it is seen that when ε is taken very small, ε^2 is as near as we wish to zero; and consequently when ε is positive and indefinitely small, ΔI is *positive.* On the other hand when ε is indefinitely small and negative, ΔI is *negative.*

Hence the total variation ΔI of the integral will be either positive or negative according as ε is positive or negative, so long as δI is different from zero; and consequently there can be neither a maximum nor a minimum value of the integral.

We know, however, if I is a maximum ΔI is always positive, and if I is a minimum ΔI is always negative; and consequently *in order to have a maximum or a minimum value of the integral δI must be zero.*

3. Applying the above result to the example given in Art. 1, we have

$$0 = 2\pi \int_{x_0}^{x_1} \left[\sqrt{1+y'^2}\, \eta + \frac{yy'}{\sqrt{1+y'^2}}\frac{d\eta}{dx} \right] dx . \tag{1}$$

Integrating by parts, the integral

$$\int_{x_0}^{x_1} \frac{yy'}{\sqrt{1+y'^2}}\, d\eta = \left[\frac{yy'}{\sqrt{1+y'^2}}\eta\right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx}\left[\frac{yy'}{\sqrt{1+y'^2}}\right]\eta dx ;$$

and since by hypothesis (p. 189), $\eta = 0$ at both of the fixed points P_0 and P_1, we have

$$\left[\frac{yy'}{\sqrt{1+y'^2}}\eta\right]_{x_0}^{x_1} = 0 .$$

Hence (1) may be written

$$0 = 2\pi \int_{x_0}^{x_1} \left[\sqrt{1+y'^2} - \frac{d}{dx}\left[\frac{yy'}{\sqrt{1+y'^2}}\right] \right] \eta dx . \tag{2}$$

4. We assert that in the expression above

$$\sqrt{1+y'^2} - \frac{d}{dx}\left[\frac{yy'}{\sqrt{1+y'^2}}\right]$$

must always be zero between the limits x_0 and x_1. For, assuming that the contrary is the case, then, since η is arbitrary, we may, with Heine, write

$$\eta = (x - x_0)(x_1 - x)\left[\sqrt{1 + y'^2} - \frac{d}{dx}\left[\frac{yy'}{\sqrt{1 + y'^2}}\right]\right],$$

where η becomes zero for the values $x = x_0$ and $x = x_1$, and is positive within the interval $x_0 \ldots x_1$.

Substituting this value of η in (2) of the preceding section, we have

$$0 = 2\pi \int_{x_0}^{x_1} \left[\sqrt{1 + y'^2} - \frac{d}{dx}\left[\frac{yy'}{\sqrt{1 + y'^2}}\right]\right]^2 (x - x_0)(x_1 - x)\, dx\,, \tag{3}$$

an expression which is positive within the whole interval $x_0 \ldots x_1$.

The integrand in (3), looked upon as a sum of infinitely small elements, has all its elements of the same sign and positive; so that the only possible way for the right hand member of (3) to be zero is that

$$\sqrt{1 + y'^2} - \frac{d}{dx}\left[\frac{yy'}{\sqrt{1 + y'^2}}\right] = 0\,.$$

We therefore have a differential equation of the second order for the determination of the unknown quantity y.

5. This differential equation is a special case of the more general differential equation, which may be derived from the integral

$$I = \int_{x_0}^{x_1} F(y, y')\, dx\,;$$

whence, as before (Arts. 1 and 3),

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right] dx$$

$$= \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]\right]\eta\, dx\,.$$

And, as above (Art. 3),

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0\,,$$

i. e.

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]. \tag{1}$$

But

$$dF(y, y') = \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial y'} dy', \tag{2}$$

or

$$dF(y, y') - \left[\frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial y'} dy'\right] = 0.$$

Hence, from (1),

$$dF(y, y') - \left[\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] dy + \frac{\partial F}{\partial y'} dy'\right] = 0$$

or

$$dF(y, y') - d\left\{y' \frac{\partial F}{\partial y'}\right\} = 0,$$

and integrating,

$$F(y, y') - y' \frac{\partial F}{\partial y'} = C, \tag{3}$$

where C is the constant of integration.

The relation (3) exists only when the integrand of the given integral does not contain *explicitly* the variable x; otherwise the relation (2) would not be true, and then we could not deduce (3).

6. Applying this relation (3) to the special case above (Art. 4), where

$$F(y, y') = y\sqrt{1 + y'^2},$$

we have

$$y\sqrt{1 + y'^2} - \frac{yy'^2}{\sqrt{1 + y'^2}} = m,$$

m being the constant of integration, a quantity which we shall consider later more in detail.

The above expression may be written

$$\frac{y(1 + y'^2 - y'^2)}{\sqrt{1 + y'^2}} = m,$$

or

$$y = m\sqrt{1 + y'^2}. \tag{I}$$

From (I) follows directly that

$$y^2 - m^2 = m^2\left[\frac{dy}{dx}\right]^2; \tag{II}$$

and (II) differentiated with respect to x, is

$$y = m^2 \frac{d^2y}{dx^2}.$$

Two solutions of this differential equation are

$$y = e^{\frac{x}{m}} \quad \text{and} \quad y = e^{-\frac{x}{m}},$$

so that the general solution is

$$y = c_1 e^{\frac{x}{m}} + c_2 e^{-\frac{x}{m}}. \tag{III}$$

It appears that we have in this expression three arbitrary constants, m, c_1, and c_2; but from (II) we have, after substituting for y^2 and $\left[\frac{dy}{dx}\right]^2$ their values from (III),

$$m^2 = 4c_1c_2.$$

Hence, writing in (III),

$$c_1 = \tfrac{1}{2}m\, e^{-\frac{x_0'}{m}} \quad \text{and} \quad c_2 = \tfrac{1}{2}m\, e^{\frac{x_0'}{m}},$$

where x_0' is a constant, we have

$$y = \tfrac{1}{2}m\,(e^{(x-x_0')/m} + e^{-(x-x_0')/m}), \tag{(III')}$$

which is the well known equation of the catenary.

The two constants x_0' and m are determined from the two conditions that the curve is to pass through the two fixed points P_0 and P_1.

REMARK. Equation (III') takes the form

$$y = \tfrac{1}{2}m\,(e^t + e^{-t}),$$

when we write $x = x_0' + mt$.

7. *Properties of the catenary.*

Equation (II) above is

$$y^2 - m^2 = m^2y'^2,$$

or

$$\pm\sqrt{y^2 - m^2} = my'.$$

Therefore

$$dx = +\frac{mdy}{\sqrt{y^2 - m^2}}, \text{ or } -\frac{mdy}{\sqrt{y^2 - m^2}}. \tag{IV}$$

The integrals of these two equations may be written

$$\left.\begin{aligned} \frac{x - x_0'}{m} &= \log\left[\frac{y + \sqrt{y^2 - m^2}}{m}\right], \\ \text{and} \quad -\frac{x - x_0}{m} &= \log\left[\frac{y - \sqrt{y^2 - m}}{m}\right]. \end{aligned}\right\} \tag{A}$$

Hence

$$\left.\begin{aligned} e^{\frac{x-x_0'}{m}} &= \frac{y+\sqrt{y^2-m^2}}{m}, \\ \text{and} \qquad e^{-\frac{x-x_0'}{m}} &= \frac{y-\sqrt{y^2-m^2}}{m}. \end{aligned}\right\} \tag{A'}$$

By the addition of equations (A′),

$$y = \tfrac{1}{2} m \left(e^{\frac{x-x_0'}{m}} + e^{-\frac{x-x_0'}{m}}\right). \tag{III'}$$

8. From (IV) we have

$$m\frac{dy}{dx} = \pm\sqrt{y^2-m^2},$$

and this from (III′) is

$$m\frac{dy}{dx} = \pm\sqrt{y^2-m^2} = \tfrac{1}{2}m\left(e^{\frac{x-x_0'}{m}} - e^{-\frac{x-x_0'}{m}}\right). \tag{V}$$

On the right hand side of this equation stands a one-valued function, but on the left hand side a two-valued function. Hence we must define the left hand expression so as to have a one-valued function.

If in the expression (V) we make $x > x_0'$, then

$$e^{\frac{x-x_0'}{m}} > e^{-\frac{x-x_0'}{m}},$$

and consequently $\sqrt{y^2-m^2}$ is positive. But when $x < x_0'$, then

$$e^{\frac{x-x_0'}{m}} < e^{-\frac{x-x_0'}{m}},$$

and then $\sqrt{y^2-m^2}$ is negative.

Hence from (V) there is only one root of the equation $dy/dx = 0$, and this is for the value $x = x_0'$.

The corresponding value of y is m. This value m is the smallest value that y can have; since $dy/dx = 0$ is the condition for maximum and minimum, and d^2y/dx^2 is positive, so that m is a minimum value of y; and since $\sqrt{y^2-m^2}$ is continuously positive or negative there is no maximum value of y.

Remark. The tangent to the curve at the point $(x = x_0', y = m)$ is parallel to the x-axis, because at this point, $dy/dx = 0$.

9. At every point of the curve we have

$$dy/dx = \tan\tau = \sqrt{y^2-m^2}/m.$$

Hence, to construct a tangent at any point of the catenary, for example at P, drop the perpendicular PQ, and describe the semi-circle on PQ as diameter.

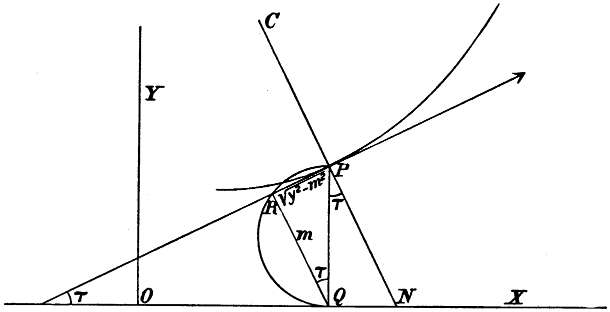

FIG. 1.

Then with radius equal to m, draw a circle from Q as centre, which cuts the semi-circle at R; join R and P. The line RP is the required tangent.

Again

$$ds^2 = dx^2 + dy^2 = \left[1 + \frac{y^2 - m^2}{m^2}\right] dx^2 = \frac{y^2 dx^2}{m^2};$$

consequently

$$ds = \frac{y dx}{m} = \tfrac{1}{2}\left(e^{(x - x_0')/m} + e^{-(x - x_0')/m}\right) dx;$$

and integrating,

$$s - s_0' = \tfrac{1}{2} m \left(e^{(x - x_0')/m} - e^{-(x - x_0')/m}\right) = \sqrt{y^2 - m^2},$$

where s_0' denotes that the arc is measured from the lowest point of the catenary.

The geometrical locus of R is a curve which cuts all the tangents to the catenary at right angles, and is therefore the *orthogonal trajectory* of this system of tangents. This trajectory has the remarkable property that the perpendiculars QR, etc., of length m, which are employed in the construction of the tangents to the catenary, are themselves tangent to the trajectory.

This trajectory possesses also the remarkable property that if we rotate it around the x-axis, the surface of rotation has a constant curvature.

Further, PN, the normal to the catenary, $= y \sec \tau = y^2/m$, and

$$\rho = \frac{\left[1 + \left[\frac{dy}{dx}\right]^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[\frac{ds}{dx}\right]^3}{\frac{d^2y}{dx^2}} = \frac{(y/m)^3}{y/m^2} = \frac{y^2}{m},$$

or

$$PN = PC \text{ (see Fig. 1)},$$

where PC is the length of the radius of curvature.

10. *The geometrical construction of the catenary.* Take an ordinate equal to $2m$. This determines the point P (see Fig. 2). With P as centre and radius

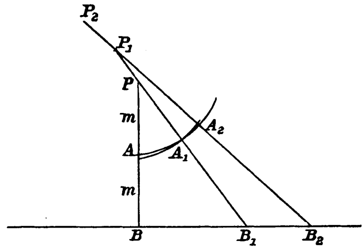

FIG. 2.

equal to m, describe a circle. This intersects PB at a point A, say. On the circumference of this circle, take a point A_1 very near A and draw the line PA_1B_1, and on this line extended take P_1 such that $P_1A_1 = A_1B_1$. With radius P_1A_1 draw another circle, and on this circle take a point A_2 very near the point A_1, and draw the line $P_1A_2B_2$. Take on this line extended the point P_2 so that $P_2A_2 = A_2B_2$, etc. The locus of the points A is the required catenary.

The accompanying figure shows approximately the relative positions of the catenary, its evolute, and the trajectory.

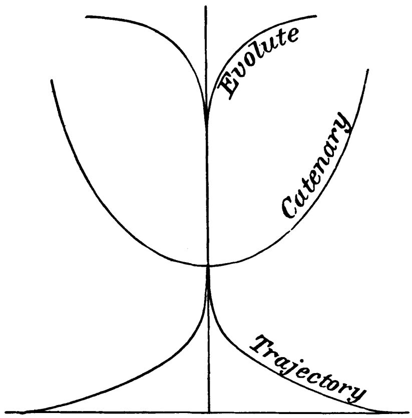

FIG. 3.

ERRATUM. In the last paragraph of Art. 2 (p. 82) the words positive and negative should be interchanged.